An Algorithmic Investigation of Hybrid Beamforming for 5G and Beyond Networks

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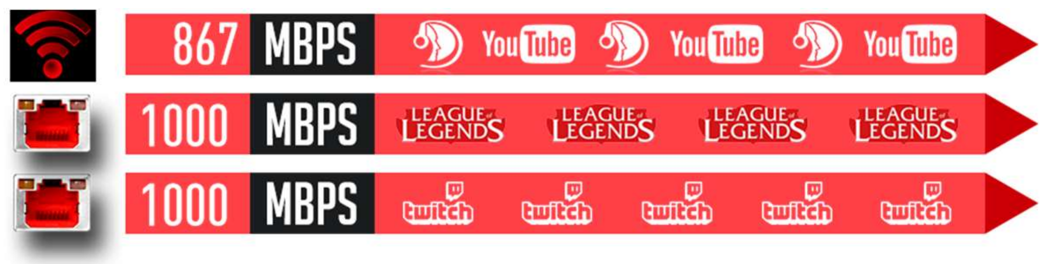
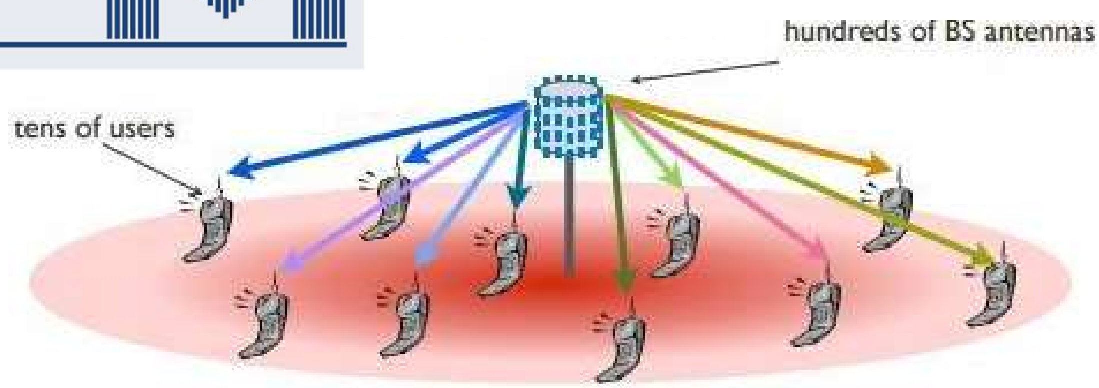
Outline

- **Background and Motivation**
- **Preliminaries of Hybrid Beamforming**
- **Hybrid Beamforming Design**
  - Improve Spectral Efficiency: Approaching the Fully Digital
  - Boost Computational Efficiency: Convex Relaxation
  - Fight for Hardware Efficiency: How Many Phase Shifters Are Needed?
- **Conclusions**

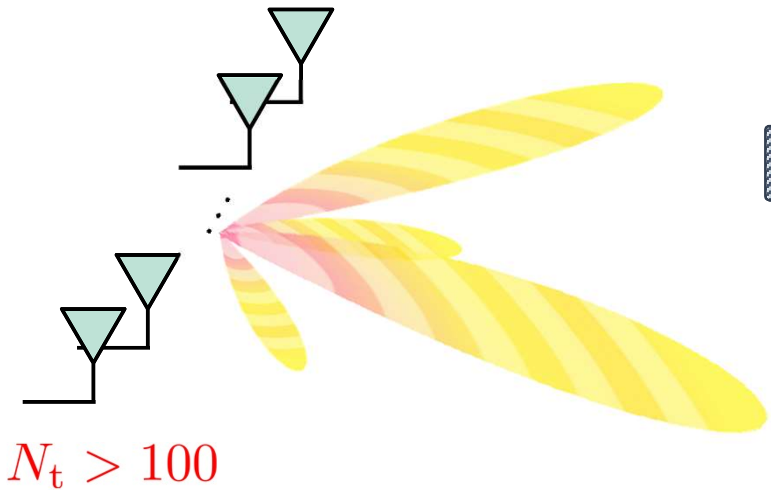
Background and Motivation

- Key enabler for 5G and beyond: Massive MIMO





Background and Motivation



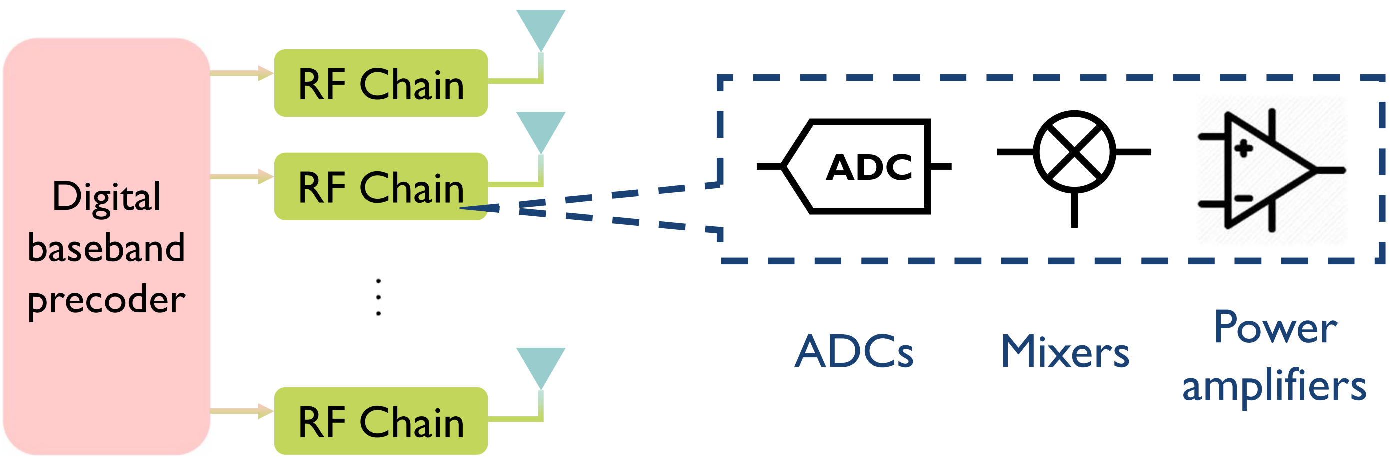
Beamforming!

Higher array gains and narrower beams

- Higher spectral efficiency
- Higher energy efficiency
- Better interference management

Background and Motivation

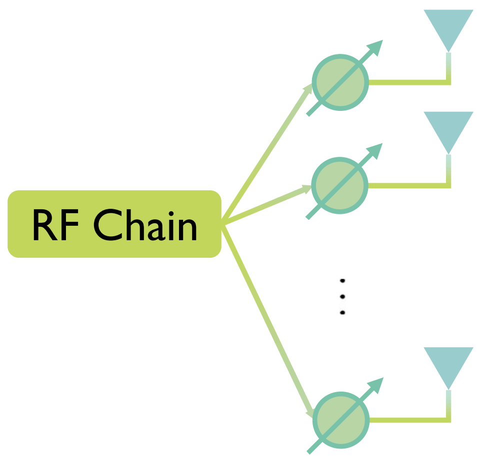
- Conventional beamforming
  - Performed digitally at the baseband
  - Requires an RF chain per antenna element



Costly and power hungry for large-scale antenna arrays

Background and Motivation

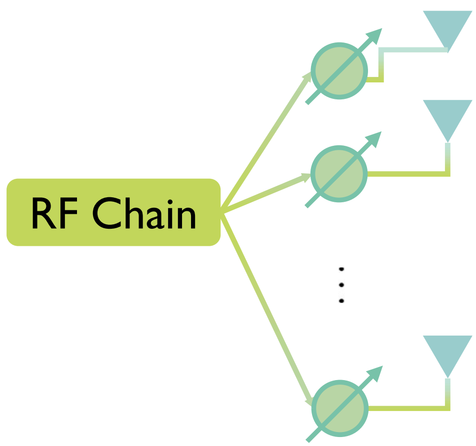
- Existing solution: Analog beamforming
  - One RF chain only



  - Beams direction readily controlled by a series of phase shifters in the RF domain
  - Low cost and low hardware complexity

Background and Motivation

- Existing solution: Analog beamforming
  - Limitations



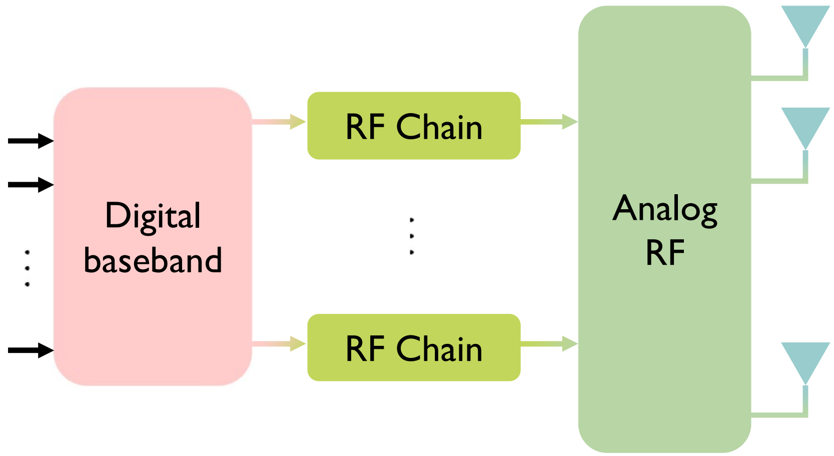
Benefits of MIMO

- Spatial multiplexing
- Support space-division multiple access (SDMA)

Analog beamforming can only support single-stream transmissions

Background and Motivation

❖ A new solution: Hybrid beamforming



➢ Multi-stream transmission, ability to support SDMA

➢ Number of RF chains much smaller than # antennas

➢ Combine the benefits of digital and analog beamforming

Background and Motivation

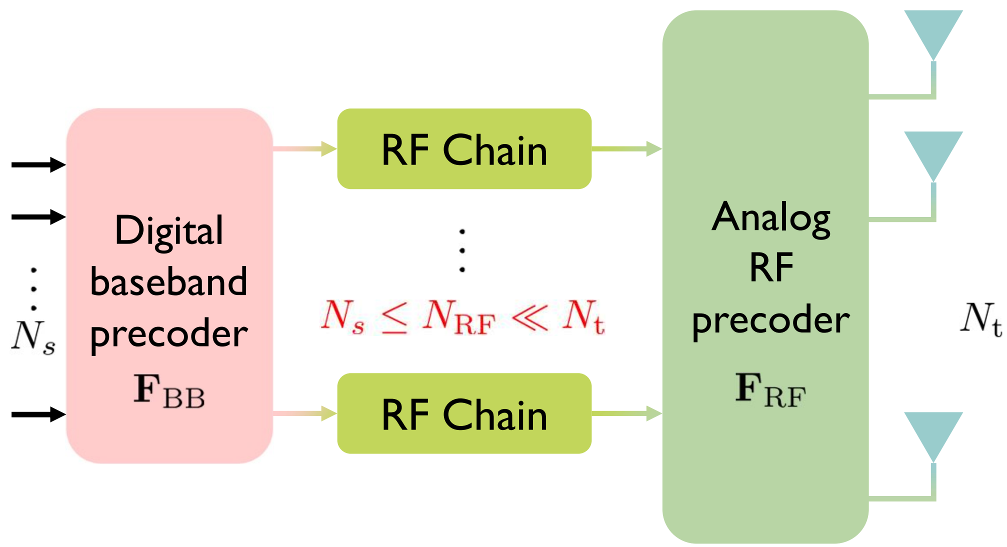
❖ Attentions on hybrid beamforming

- O. E. Ayach, S. Rajagopal, S. Abu-Surra, Z. Pi, and R. W. Heath, Jr., "Spatially sparse precoding in millimeter wave MIMO systems," *IEEE Trans. Wireless Commun.*, vol. 13, no. 3, pp. 1499-1513, Mar. 2014.
    - **The 2017 Marconi Prize Paper Award in Wireless Communications**
- F. Sohrabi and W. Yu, "Hybrid digital and analog beamforming design for large-scale antenna arrays," *IEEE J. Sel. Topics Signal Process.*, vol. 10, no. 3, pp. 501-513, Apr. 2016.
    - **The 2017 IEEE Signal Processing Society Best Paper Award**
- A. Alkhateeb, O. El Ayach, G. Leus, and R. W. Heath, Jr., "Channel estimation and hybrid precoding for millimeter wave cellular systems," *IEEE J. Sel. Topics Signal Process.*, vol. 8, no. 5, pp. 831-846, Oct. 2014.
    - **The 2016 Signal Processing Society Young Author Best Paper Award**
- X. Yu, J.-C. Shen, J. Zhang, and K. B. Letaief, "Alternating minimization algorithms for hybrid precoding in millimeter wave MIMO systems," *IEEE J. Sel. Topics Signal Process.*, vol. 10, no. 3, pp. 485-500, Apr. 2016.
    - **The 2018 Signal Processing Society Young Author Best Paper Award**

Preliminaries of Hybrid Beamforming

Preliminaries of Hybrid Beamforming

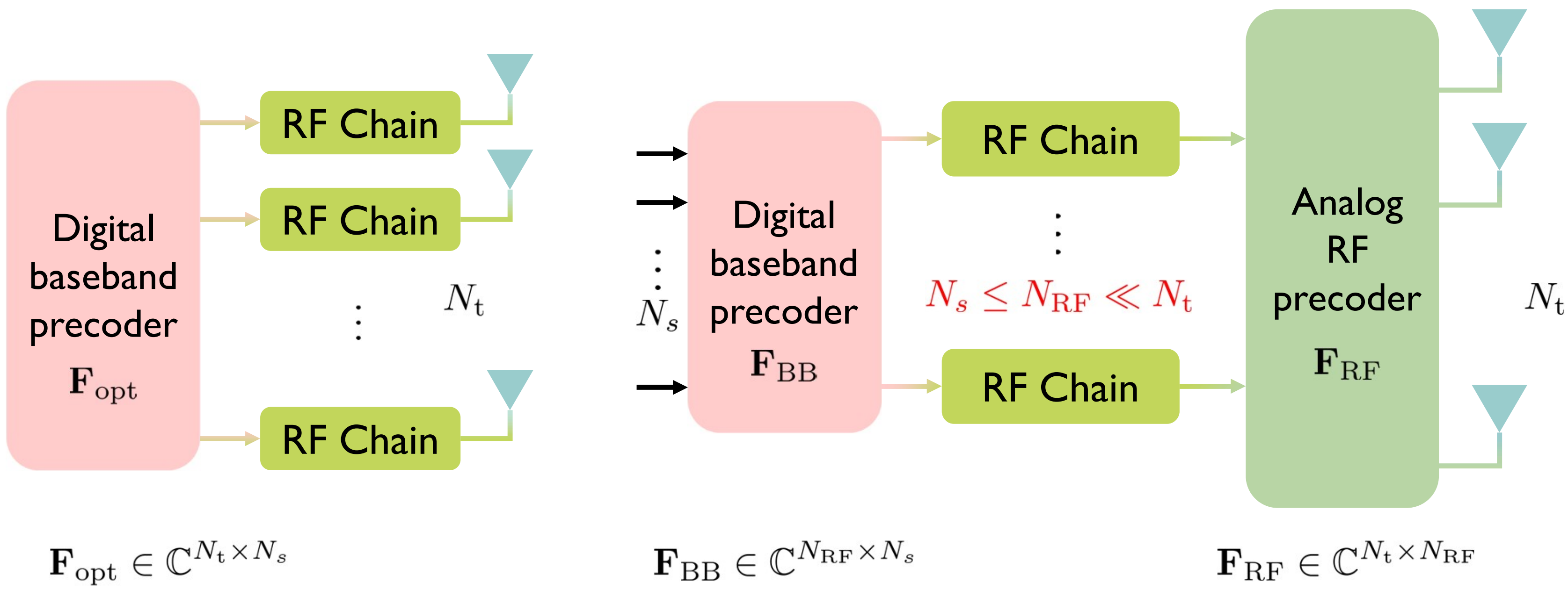
- Hybrid beamforming
  - Also called *Hybrid precoding; Analog/digital precoding*
  - Notations in hybrid beamforming



$\mathbf{F}_{\mathrm{BB}} \in \mathbb{C}^{N_{\mathrm{RF}} \times N_s}$ $\mathbf{F}_{\mathrm{RF}} \in \mathbb{C}^{N_{\mathrm{t}} \times N_{\mathrm{RF}}}$

Preliminaries of Hybrid Beamforming

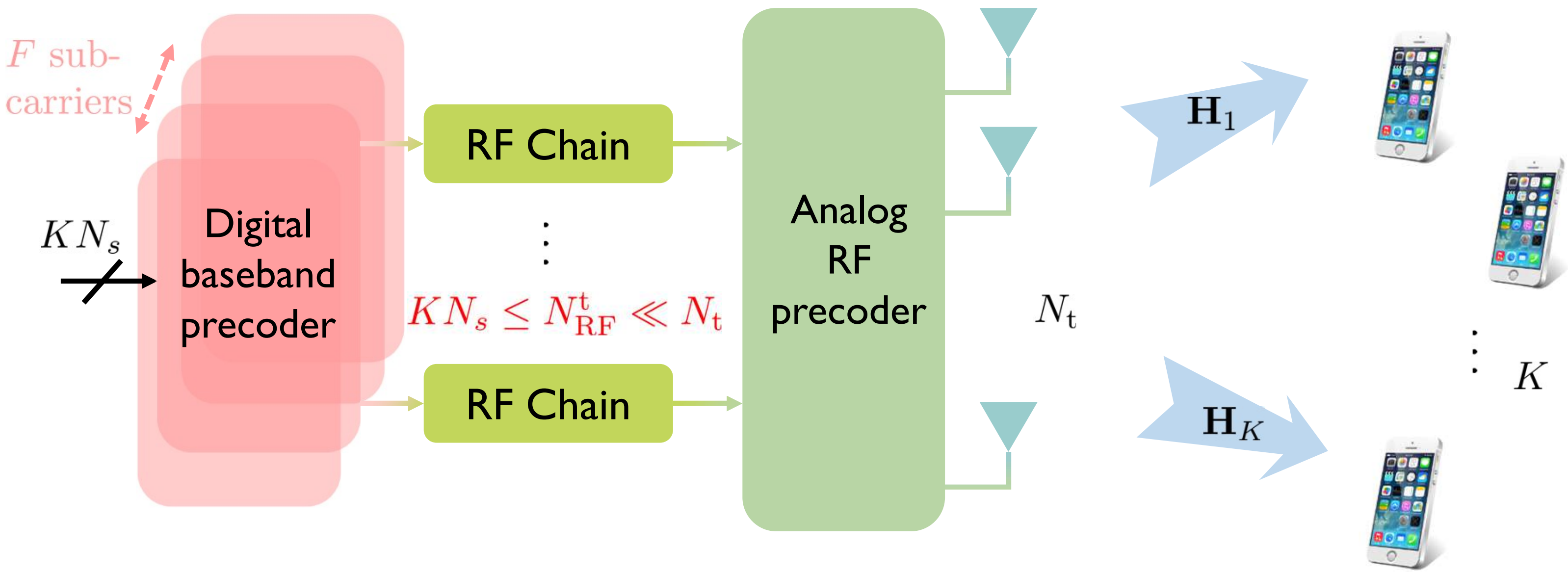
- Fully digital precoding vs. Hybrid precoding



- Main differentiating part: Analog RF precoder
- Mapping from low-dimensional RF chains to high-dimensional antennas, typically implemented by phase shifters

Preliminaries of Hybrid Beamforming

- General multiuser multicarrier (MU-MC) systems



- One separate digital precoder for each user on each subcarrier $\mathbf{F}_{\mathrm{BB}k,f}$
- Analog precoder $\mathbf{F}_{\mathrm{RF}}$ is shared by all the users and subcarriers

Preliminaries of Hybrid Beamforming

- Generic hybrid beamforming problem
  - Minimize the Euclidean distance between the hybrid precoders and the fully digital precoder [O. El Ayach *et al.*, 2014]

$$\begin{aligned} \underset{\mathbf{F}_{\mathrm{RF}},\mathbf{F}_{\mathrm{BB}}}{\text{minimize}} \quad & \|\mathbf{F}_{\mathrm{opt}} - \mathbf{F}_{\mathrm{RF}}\mathbf{F}_{\mathrm{BB}}\|_F^2 \\ \text{subject to} \quad & \|\mathbf{F}_{\mathrm{RF}}\mathbf{F}_{\mathrm{BB}}\|_F^2 \leq P_{\max} \\ & \mathbf{F}_{\mathrm{RF}} \in \mathcal{A}_x \end{aligned}$$

  - This formulation applies with an arbitrary digital precoder.
  - It is applicable to different hybrid beamforming structures.
  - It facilitates beamforming algorithm design.
  - The obtained algorithmic approaches also help other formulations.

Preliminaries of Hybrid Beamforming

❖ Generic hybrid beamforming problem

$$\begin{aligned} \underset{\mathbf{F}_{\mathrm{RF}},\mathbf{F}_{\mathrm{BB}}}{\text{minimize}} \quad & \|\mathbf{F}_{\mathrm{opt}} - \mathbf{F}_{\mathrm{RF}}\mathbf{F}_{\mathrm{BB}}\|_F^2 \\ \text{subject to} \quad & \|\mathbf{F}_{\mathrm{RF}}\mathbf{F}_{\mathrm{BB}}\|_F^2 \le P_{\max} \\ & \mathbf{F}_{\mathrm{RF}} \in \mathcal{A}_x \end{aligned}$$

Main difficulties

- Unit modulus constraints for phases $|(\mathbf{F}_{\mathrm{RF}})_{i,j}| = 1$
- Structure constraints for $\mathcal{A}_x$ (different hybrid architectures)

Preliminaries of Hybrid Beamforming

❖ Unit modulus constraints

Common approaches

➢ Codebook based, e.g., OMP [O. El Ayach *et al.*, 2014]

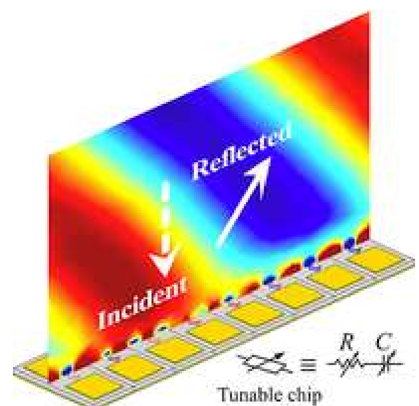
The columns of the analog precoding matrix $\mathbf{F}_{RF}$ selected from array response vectors

➢ Manifold optimization – directly tackle unit modulus constraints [Yu *et al.*, 2016]

➢ Convex relaxation [Yu *et al.*, 2019]

Other applications

➢ Intelligent reflecting surfaces (IRSs)



[Liu *et al.*, 2019]

Preliminaries of Hybrid Beamforming

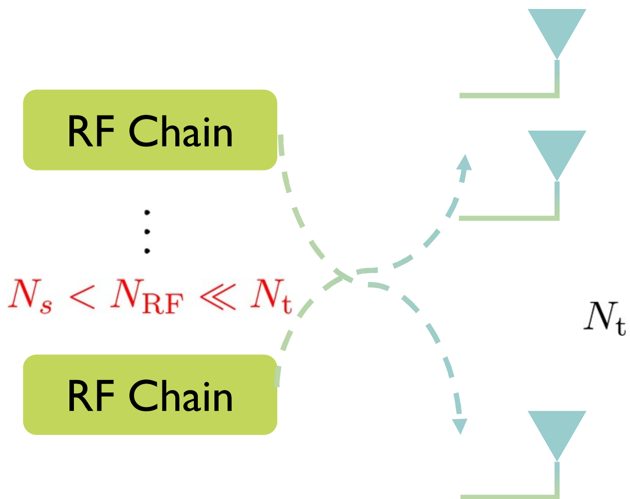
❖ A taxonomy of hybrid beamforming structures

(I) Mapping strategy:

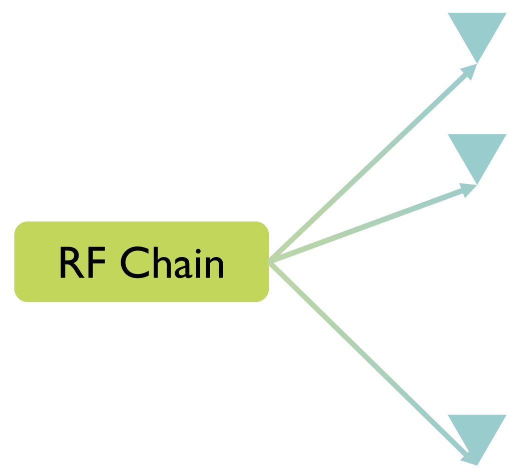
Which antennas should be connected to each RF chain?

(II) Hardware implementation:

What kind of hardware should be used to realize each connection?



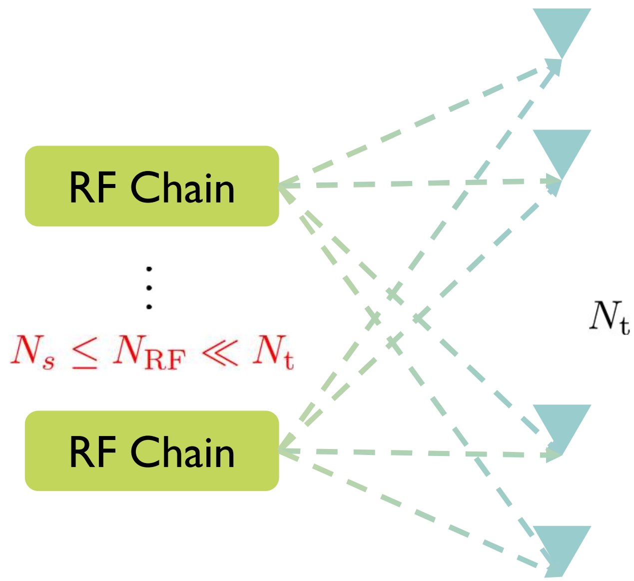
Signal flow



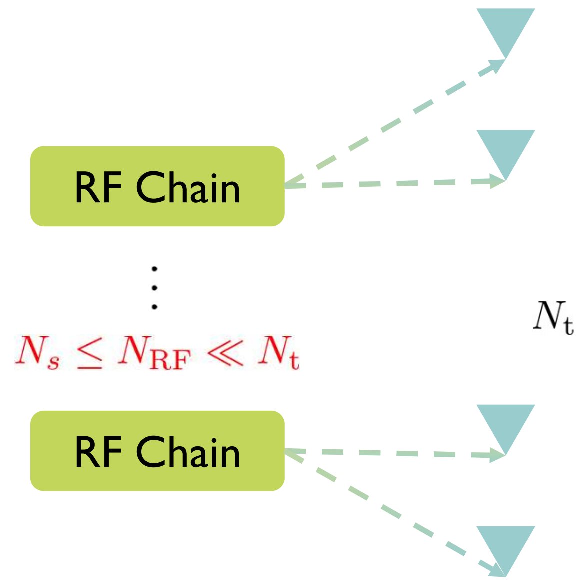
Adopted hardware

Preliminaries of Hybrid Beamforming

- The state-of-the-art hybrid beamforming structures
  - Mainly focus on different mapping strategies



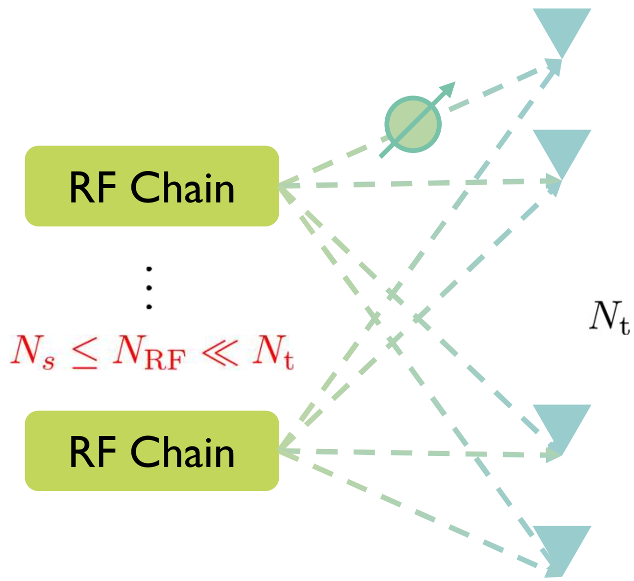
Fully-connected



Partially-connected

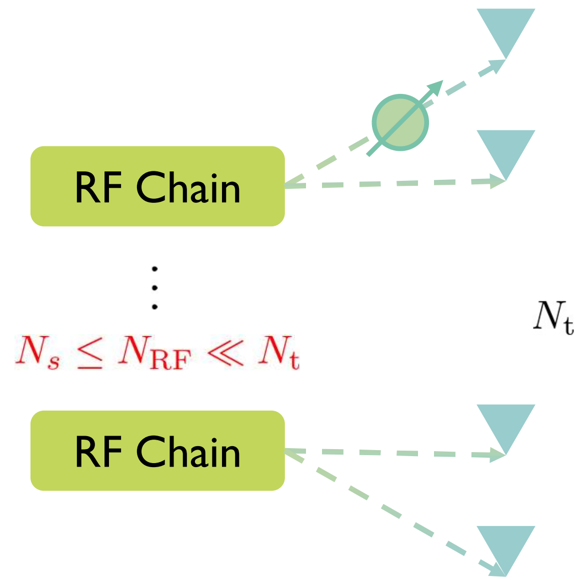
Preliminaries of Hybrid Beamforming

- The state-of-the-art hybrid beamforming structures
  - One prevalent hardware implementation: Single phase shifter (SPS)



SPS Fully-connected

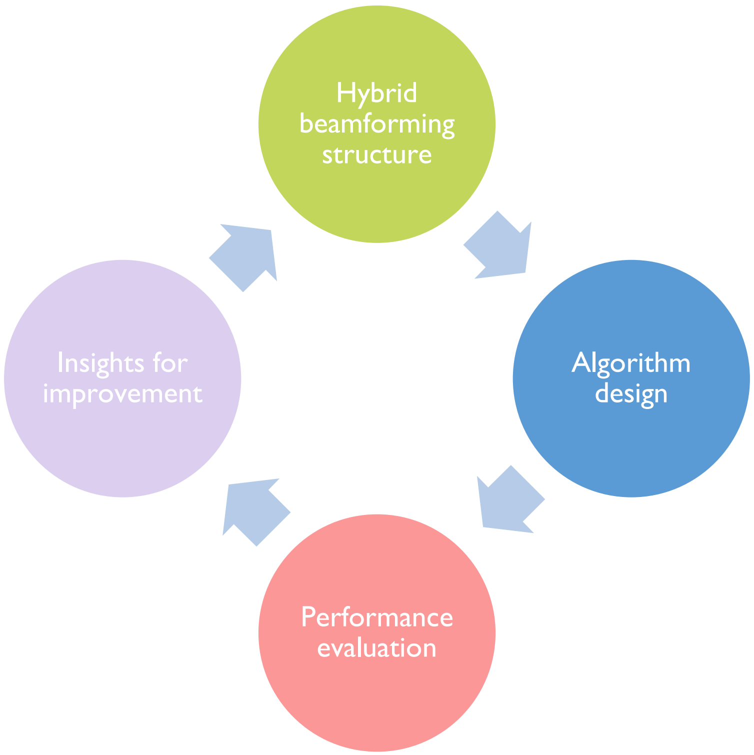
$N_{\mathrm{PS}} = N_{\mathrm{t}} N_{\mathrm{RF}}$

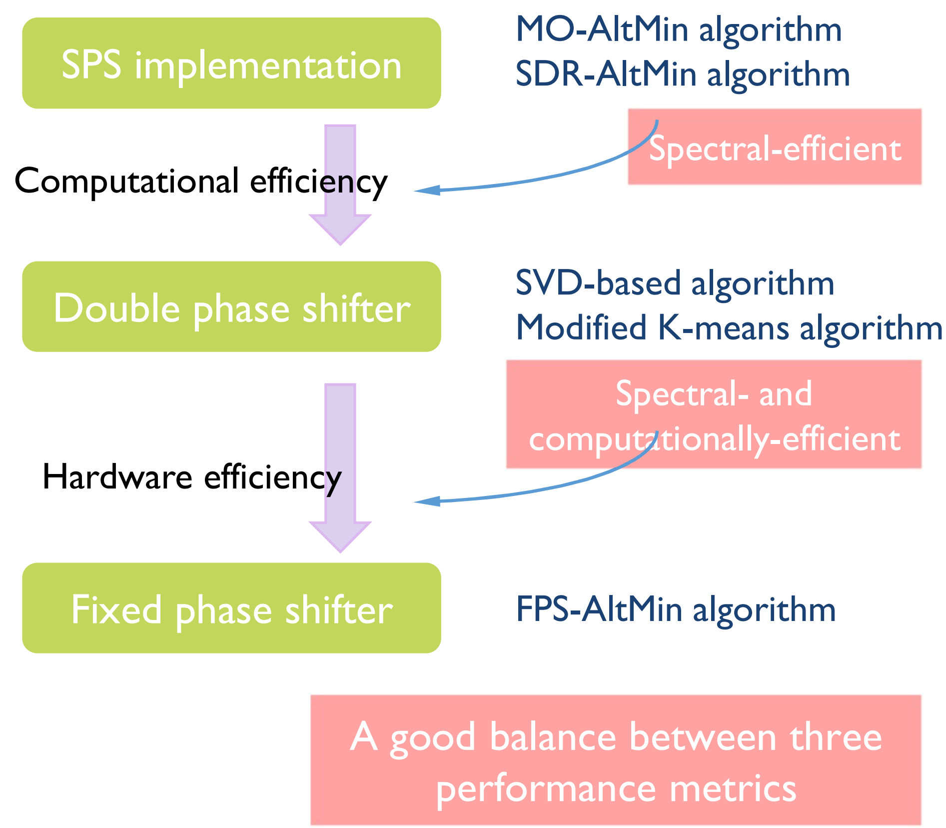


SPS Partially-connected

$N_{\mathrm{PS}} = N_{\mathrm{t}}$

Preliminaries of Hybrid Beamforming





Effective algorithms are required to reveal system insights

Preliminaries of Hybrid Beamforming

- Three key aspects to investigate

Spectral efficiency

- **Q1:** Can hybrid beamforming provide performance close to the fully digital one?

Hardware efficiency

- **Q2:** How many RF chains are needed?
- **Q3:** How many phase shifters are needed?

Computational efficiency

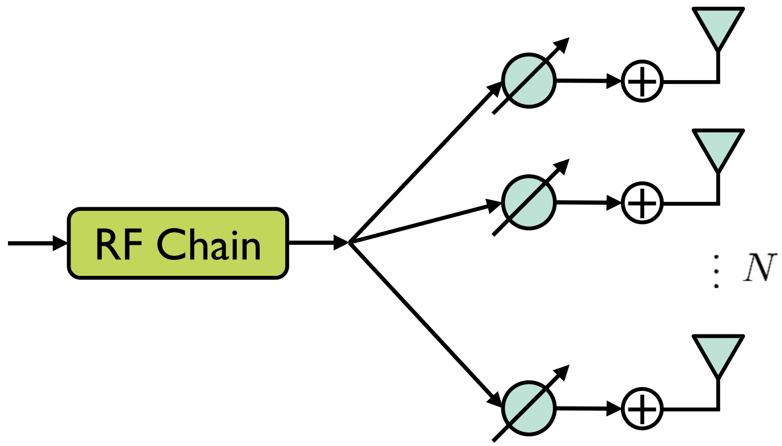
- **Q4:** How to efficiently design hybrid beamforming algorithms?

Improve Spectral Efficiency: Approaching the Fully Digital Beamforming

[Ref] X. Yu, J.-C. Shen, J. Zhang, and K. B. Letaief, "Alternating minimization algorithms for hybrid precoding in millimeter wave MIMO systems," *IEEE J. Sel. Topics Signal Process., Special Issue on Signal Process. for Millimeter Wave Wireless Commun.*, vol. 10, no. 3, pp. 485-500, Apr. 2016. (**The 2018 IEEE Signal Processing Society Young Author Best Paper Award**)

Improve Spectral Efficiency

- Single phase shifter (SPS) implementation

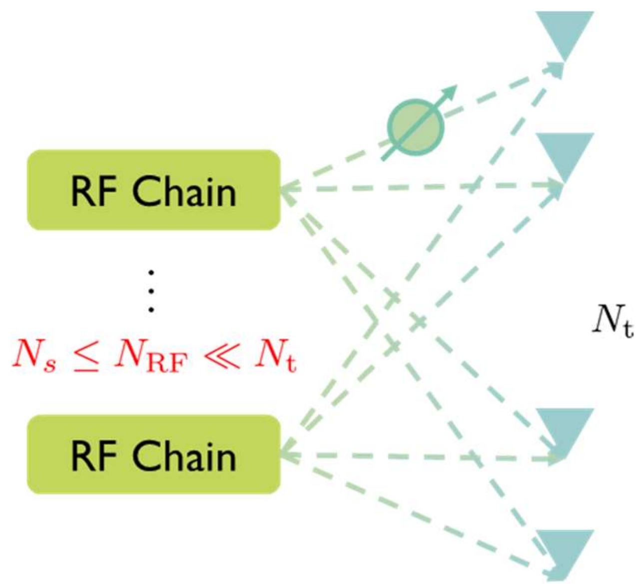


$$N = \begin{cases} N_t & \text{fully-connected} \\ N_t / N_{RF}^t & \text{partially-connected} \end{cases}$$

- Fully digital achieving condition: $N_{RF}^t \geq 2KN_s,\ N_{RF}^r \geq 2N_s$

Q: Can we further reduce the number of RF chains?

(I) Fully-Connected Mapping



Improve Spectral Efficiency

(I) Fully-Connected Mapping

- Start from single-user systems

$$\begin{aligned} \underset{\mathbf{F}_{\mathrm{RF}},\mathbf{F}_{\mathrm{BB}}}{\text{minimize}} \quad & \|\mathbf{F}_{\mathrm{opt}} - \mathbf{F}_{\mathrm{RF}}\mathbf{F}_{\mathrm{BB}}\|_F^2 \\ \text{subject to} \quad & |(\mathbf{F}_{\mathrm{RF}})_{i,j}| = 1, \forall i, j. \end{aligned}$$

  - Alternating minimization

$$\underset{\mathbf{F}_{\mathrm{BB}}}{\text{minimize}} \quad \|\mathbf{F}_{\mathrm{opt}} - \mathbf{F}_{\mathrm{RF}}\mathbf{F}_{\mathrm{BB}}\|_F^2$$

$$\begin{aligned} \underset{\mathbf{F}_{\mathrm{RF}}}{\text{minimize}} \quad & \|\mathbf{F}_{\mathrm{opt}} - \mathbf{F}_{\mathrm{RF}}\mathbf{F}_{\mathrm{BB}}\|_F^2 \\ \text{subject to} \quad & |(\mathbf{F}_{\mathrm{RF}})_{i,j}| = 1, \forall i, j. \end{aligned}$$

  - Digital precoder: $\mathbf{F}_{\mathrm{BB}} = \mathbf{F}_{\mathrm{RF}}^{\dagger}\mathbf{F}_{\mathrm{opt}}$
  - Difficulty: Analog precoder design with the unit modulus constraints

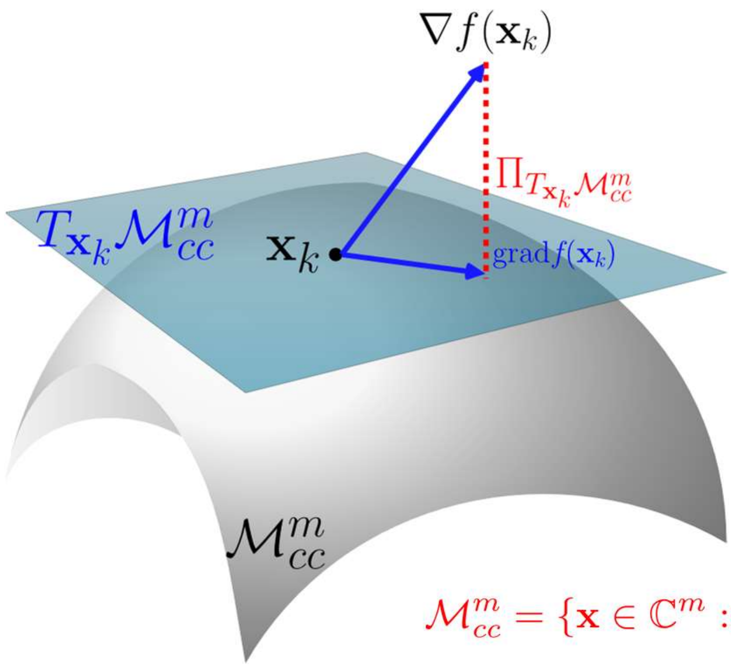
$$\begin{aligned} \underset{\mathbf{F}_{\mathrm{RF}}}{\text{minimize}} \quad & \|\mathbf{F}_{\mathrm{opt}} - \mathbf{F}_{\mathrm{RF}}\mathbf{F}_{\mathrm{BB}}\|_F^2 \\ \text{subject to} \quad & |(\mathbf{F}_{\mathrm{RF}})_{i,j}| = 1, \forall i, j. \end{aligned}$$

  - The vector $\mathbf{x} = \mathrm{vec}(\mathbf{F}_{\mathrm{RF}})$ forms a complex circle manifold

$$\mathcal{M}_{cc}^m = \{\mathbf{x} \in \mathbb{C}^m : |\mathbf{x}_1| = |\mathbf{x}_2| = \cdots = |\mathbf{x}_m| = 1\}, \quad m = N_{\mathrm{t}} N_{\mathrm{RF}}^{\mathrm{t}}.$$

Improve Spectral Efficiency
(I) Fully-Connected Mapping

- Manifold optimization (cont.)
  - Euclidean space: gradient descent
  - Similar approaches on manifolds?

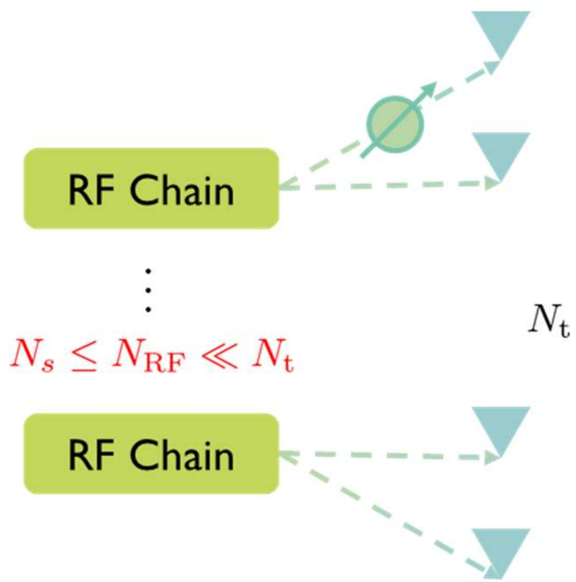


- Tangent space
- Riemannian gradient
- Retraction

Gradient-based algorithm on manifolds

$$\mathcal{M}_{cc}^m = \{\mathbf{x} \in \mathbb{C}^m : |\mathbf{x}_1| = |\mathbf{x}_2| = \cdots = |\mathbf{x}_m| = 1\}, \quad m = N_t N_{RF}^t.$$

(II) Partially-Connected Mapping



Improve Spectral Efficiency

(II) Partially-Connected Mapping

- SPS partially-connected
  - $\mathcal{A}_x$: Block diagonal $\mathbf{F}_{\mathrm{RF}}$ with unit modulus non-zero elements

$$\mathbf{F}_{\mathrm{RF}} = \begin{bmatrix} \mathbf{p}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{p}_2 & & \mathbf{0} \\ \vdots & & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{p}_{N_{\mathrm{RF}}^{\mathrm{t}}} \end{bmatrix} \qquad \mathbf{p}_i = \left[\exp\left(\jmath\theta_{(i-1)\frac{N_t}{N_{\mathrm{RF}}^t}+1}\right), \cdots, \exp\left(\jmath\theta_{i\frac{N_t}{N_{\mathrm{RF}}^t}}\right)\right]^T$$

phase shifters connected to the *i*-th RF chain

  - Problem decoupled for each RF chain
  - Closed-form solution for $\mathbf{F}_{\mathrm{RF}}$

$$\arg\{(\mathbf{F}_{\mathrm{RF}})_{i,l}\} = \arg\left\{(\mathbf{F}_{\mathrm{opt}})_{i,:}(\mathbf{F}_{\mathrm{BB}})_{l,:}{}^H\right\}, \quad 1 \le i \le N_t,\ l = \left\lceil i\frac{N_{\mathrm{RF}}^t}{N_t}\right\rceil$$

Improve Spectral Efficiency

(II) Partially-Connected Mapping

- SPS partially-connected (cont.)
  - Optimization of $\mathbf{F}_{\mathrm{BB}}$

$$\begin{aligned} \underset{\mathbf{F}_{\mathrm{BB}}}{\text{minimize}} \quad & \|\mathbf{F}_{\mathrm{opt}} - \mathbf{F}_{\mathrm{RF}}\mathbf{F}_{\mathrm{BB}}\|_F^2 \\ \text{subject to} \quad & \|\mathbf{F}_{\mathrm{BB}}\|_F^2 = \frac{N_{\mathrm{RF}}^{\mathrm{t}} N_s}{N_{\mathrm{t}}}. \end{aligned}$$

  - Reformulate as a non-convex problem

$$\begin{aligned} \underset{\mathbf{Y}\in\mathbb{H}^n}{\text{minimize}} \quad & \mathrm{Tr}(\mathbf{CY}) \\ \text{subject to} \quad & \begin{cases} \mathrm{Tr}(\mathbf{A}_1\mathbf{Y}) = \frac{N_{\mathrm{RF}}^{\mathrm{t}} N_s}{N_{\mathrm{t}}} \\ \mathrm{Tr}(\mathbf{A}_2\mathbf{Y}) = 1 \\ \mathbf{Y} \succeq 0, \ \mathrm{rank}(\mathbf{Y}) = 1 \end{cases} \end{aligned}$$

convex

$$n = N_{\mathrm{RF}}^t N_s + 1, \ \mathbf{y} = \begin{bmatrix} \mathrm{vec}(\mathbf{F}_{\mathrm{BB}}) & t \end{bmatrix}^T,$$

$$\mathbf{Y} = \mathbf{y}\mathbf{y}^H, \ \mathbf{f} = \mathrm{vec}(\mathbf{F}_{\mathrm{opt}}),$$

$$\mathbf{A}_1 = \begin{bmatrix} \mathbf{I}_{n-1} & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix}, \mathbf{A}_2 = \begin{bmatrix} \mathbf{0}_{n-1} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix},$$

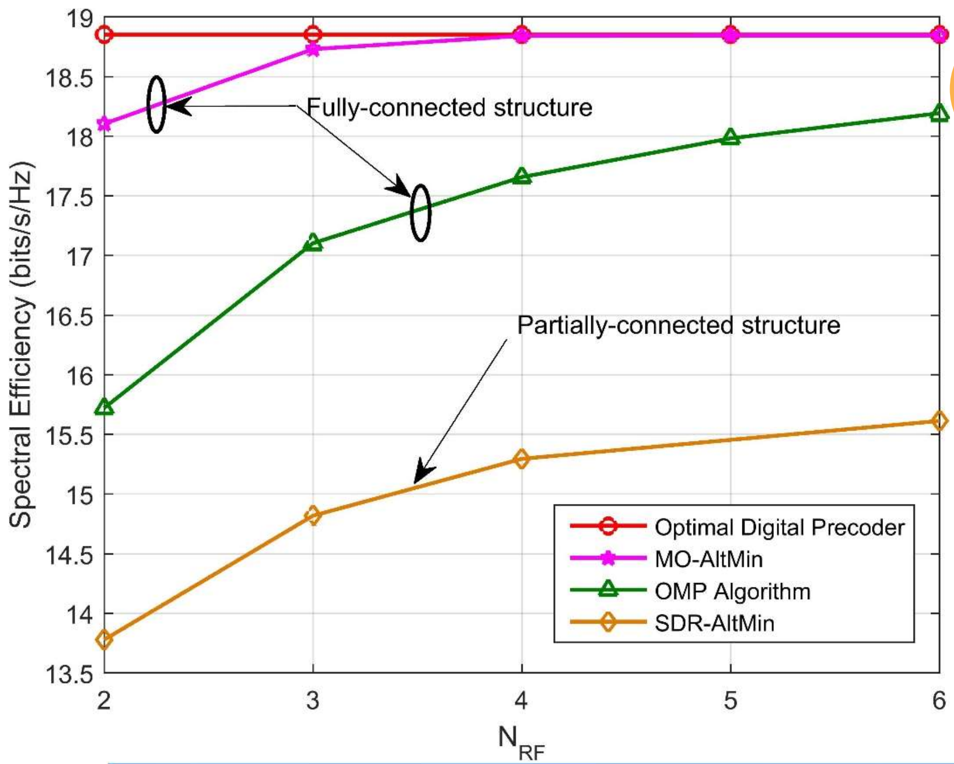
$$\mathbf{C} = \begin{bmatrix} (\mathbf{I}_{N_s} \otimes \mathbf{F}_{\mathrm{RF}})^H (\mathbf{I}_{N_s} \otimes \mathbf{F}_{\mathrm{RF}}) & -(\mathbf{I}_{N_s} \otimes \mathbf{F}_{\mathrm{RF}})^H \mathbf{f} \\ -\mathbf{f}^H (\mathbf{I}_{N_s} \otimes \mathbf{F}_{\mathrm{RF}}) & \mathbf{f}^H\mathbf{f} \end{bmatrix}.$$

  - Semidefinite relaxation (SDR) is tight for this case so globally optimal solution is obtained [Z.-Q. Luo *et al.*, 2010]

Improve Spectral Efficiency

❖ Simulation results

$N_t = 144,\ N_r = 36,\ N_{RF}^t = N_{RF}^r = N_{RF},\ N_s = 2,\ \text{SNR} = 0\,\text{dB}$



- ~N_s RF chains are sufficient for the fully-connected mapping
- Employing fewer PSs, the partially-connected mapping needs more RF chains

Limitation: Computational efficiency of the MO-AltMin is not good, thus difficult to extend to MU-MC settings

Boost Computational Efficiency: Convex Relaxation

[Ref] X. Yu, J. Zhang, and K. B. Letaief, "Alternating minimization for hybrid precoding in multiuser OFDM mmWave Systems," in *Proc. Asilomar Conf. on Signals, Systems, and Computers*, Pacific Grove, CA, Nov. 2016. **(Invited Paper)**

[Ref] X. Yu, J. Zhang, and K. B. Letaief, "Doubling phase shifters for efficient hybrid precoding in millimeter-wave multiuser OFDM systems," *J. Commun. Inf. Netw.*, vol. 4, no. 2, pp. 51-67, Jul. 2019.

Boost Computational Efficiency

- Main difficulty in designing the SPS implementation
  - Analog precoder with the unit modulus constraints

$$\begin{aligned} \underset{\mathbf{F}_{\mathrm{RF}},\mathbf{F}_{\mathrm{BB}}}{\text{minimize}} \quad & \|\mathbf{F}_{\mathrm{opt}} - \mathbf{F}_{\mathrm{RF}}\mathbf{F}_{\mathrm{BB}}\|_F^2 \\ \text{subject to} \quad & |(\mathbf{F}_{\mathrm{RF}})_{i,j}| = 1, \forall i, j. \end{aligned}$$

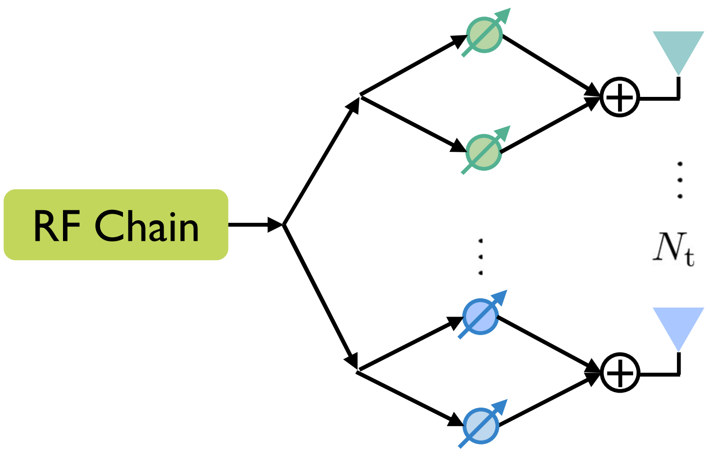
  - An intuitive way to boost computational efficiency is to relax this highly non-convex constraint as a convex one

$$\begin{aligned} \underset{\mathbf{F}_{\mathrm{RF}},\mathbf{F}_{\mathrm{BB}}}{\text{minimize}} \quad & \|\mathbf{F}_{\mathrm{opt}} - \mathbf{F}_{\mathrm{RF}}\mathbf{F}_{\mathrm{BB}}\|_F^2 \\ \text{subject to} \quad & |(\mathbf{F}_{\mathrm{RF}})_{i,j}| \leq \gamma, \forall i, j. \end{aligned}$$

  - The value of γ does not affect the hybrid beamformer design
  - We shall choose γ=2 instead of keeping it as 1. Why?

Boost Computational Efficiency

- Double phase shifter (DPS) implementation
  - The relaxed solution with γ=2 can be realized by a hardware implementation



- Unit modulus constraint is eliminated
- Sum of two phase shifters

$$|e^{\jmath\theta_1} + e^{\jmath\theta_2}| \leq 2$$

Boost Computational Efficiency

(I) Fully-Connected Mapping

- Fully-connected mapping
  - RF-only precoding

$$\begin{aligned} \underset{\mathbf{F}_{\mathrm{RF}}}{\text{minimize}} \quad & \|\mathbf{F}_{\mathrm{opt}} - \mathbf{F}_{\mathrm{RF}}\mathbf{F}_{\mathrm{BB}}\|_F^2 \\ \text{subject to} \quad & |(\mathbf{F}_{\mathrm{RF}})_{i,j}| \le 2 \end{aligned} \iff \underset{\mathbf{x}}{\text{minimize}} \quad \frac{1}{2}\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2 + 2\|\mathbf{x}\|_1$$

LASSO

  - Closed-form solution for semi-unitary codebooks $\mathbf{F}_{\mathrm{BB}}\mathbf{F}_{\mathrm{BB}}^H = \mathbf{I}_{N_{\mathrm{RF}}^{\mathrm{t}}}$

$$\mathbf{F}_{\mathrm{RF}}^{\star} = \mathbf{F}_{\mathrm{opt}}\mathbf{F}_{\mathrm{BB}}^H - \exp\left\{j\angle\left(\mathbf{F}_{\mathrm{opt}}\mathbf{F}_{\mathrm{BB}}^H\right)\right\} \circ \left(\left|\mathbf{F}_{\mathrm{opt}}\mathbf{F}_{\mathrm{BB}}^H\right| - 2\right)^+ .$$

  - Hybrid precoding

$$\begin{aligned} \underset{\mathbf{F}_{\mathrm{RF}},\mathbf{F}_{\mathrm{BB}}}{\text{minimize}} \quad & \|\mathbf{F}_{\mathrm{opt}} - \mathbf{F}_{\mathrm{RF}}\mathbf{F}_{\mathrm{BB}}\|_F^2 \\ \text{subject to} \quad & |(\mathbf{F}_{\mathrm{RF}})_{i,j}| \le 2 \end{aligned} \Rightarrow \text{Matrix factorization}$$

Redundant

Boost Computational Efficiency

(I) Fully-Connected Mapping

- Fully-connected mapping (cont.)
  - Optimality in single-carrier systems

    $\mathbf{F}_{\text{opt}} = \mathbf{F}_{\text{RF}}\mathbf{F}_{\text{BB}}$ with $N_{\text{RF}}^{\text{t}} = KN_s$ and $N_{\text{RF}}^{\text{r}} = N_s$ when $F = 1$

    Minimum number of RF chains

  - It reduces the required number of RF chains by half for achieving the fully digital precoding

  - Multi-carrier systems

$$\underset{\mathbf{F}_{\text{RF}},\mathbf{F}_{\text{BB}}}{\text{minimize}} \quad \|\mathbf{F}_{\text{opt}} - \mathbf{F}_{\text{RF}}\mathbf{F}_{\text{BB}}\|_F^2$$

  - Low-rank matrix approximation: SVD, globally optimal solution

Boost Computational Efficiency

(I) Fully-Connected Mapping

- Fully-connected mapping (cont.)
  - Q: How to use this relaxed result for SPS implementation?
  - Optimal solution:

$$\mathbf{F}_{\mathrm{RF}}\mathbf{F}_{\mathrm{BB}} = \mathbf{U}_1\mathbf{S}_1\mathbf{V}_1^H$$

  - Some clues: The unitary matrix $\mathbf{U}_1$ fully extracts the information of the column space of $\mathbf{F}_{\mathrm{RF}}\mathbf{F}_{\mathrm{BB}}$, whose basis are the orthonormal columns in $\mathbf{F}_{\mathrm{RF}}$
  - Phase extraction

$$\mathbf{F}_{\mathrm{RF}} = \exp\{\jmath\angle(\mathbf{U}_1)\}, \quad \mathbf{F}_{\mathrm{BB}} = \mathbf{S}_1\mathbf{V}_1^H$$

unit modulus constraint

Convex relaxation-enabled (CR-enabled) SPS

Boost Computational Efficiency

(II) Partially-Connected Mapping

- Partially-connected mapping
  - Block diagonal structure

$$\mathbf{F}_{\mathrm{RF}} = \begin{bmatrix} \mathbf{p}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{p}_2 & & \mathbf{0} \\ \vdots & & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{p}_{N_{\mathrm{RF}}^{\mathrm{t}}} \end{bmatrix} \qquad \mathbf{p}_j = \left[a_{(j-1)\frac{N_{\mathrm{t}}}{N_{\mathrm{RF}}^{\mathrm{t}}}+1}, \cdots, a_{j\frac{N_{\mathrm{t}}}{N_{\mathrm{RF}}^{\mathrm{t}}}} \right]^T$$

  - Decoupled for each RF chain

$$\mathcal{P}_j : \quad \underset{\{a_i\},\mathbf{x}_j}{\text{minimize}} \sum_{i \in \mathcal{F}_j} \|\mathbf{y}_i - a_i \mathbf{x}_j\|_2^2 ,$$

$$\mathcal{F}_j = \left\{ i \in \mathbb{Z} \,\middle|\, (j-1)\frac{N_{\mathrm{t}}}{N_{\mathrm{RF}}^{\mathrm{t}}} + 1 \leq i \leq j\frac{N_{\mathrm{t}}}{N_{\mathrm{RF}}^{\mathrm{t}}} \right\}, \mathbf{y}_i = \mathbf{F}_{\mathrm{opt}}^T(i,:), \text{ and } \mathbf{x}_j = \mathbf{F}_{\mathrm{BB}}^T(j,:)$$

  - Eigenvalue problem $\mathbf{x}_j^\star = \boldsymbol{\lambda}_1 \left(\sum_{i \in \mathcal{F}_j} \mathbf{y}_i \mathbf{y}_i^H \right), \quad a_i^\star = \frac{\mathbf{x}_j^H \mathbf{y}_i}{||\mathbf{x}_j||_2^2}$

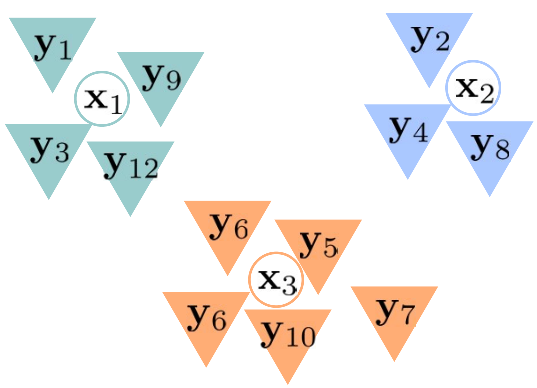
Boost Computational Efficiency

(II) Partially-Connected Mapping

- DPS partially-connected mapping (cont.)
  - Not much performance gain obtained by simply adopting the DPS implementation
  - Dynamic mapping:

    Adaptively separate all N_{t} antennas into N_{RF} groups

$$\underset{\{\mathcal{D}_j\}_{j=1}^{N_{\mathrm{RF}}^{\mathrm{t}}}}{\text{maximize}} \quad \sum_{j=1}^{N_{\mathrm{RF}}^{\mathrm{t}}} \lambda_1 \left(\sum_{i \in \mathcal{D}_j} \mathbf{y}_i \mathbf{y}_i^H \right)$$

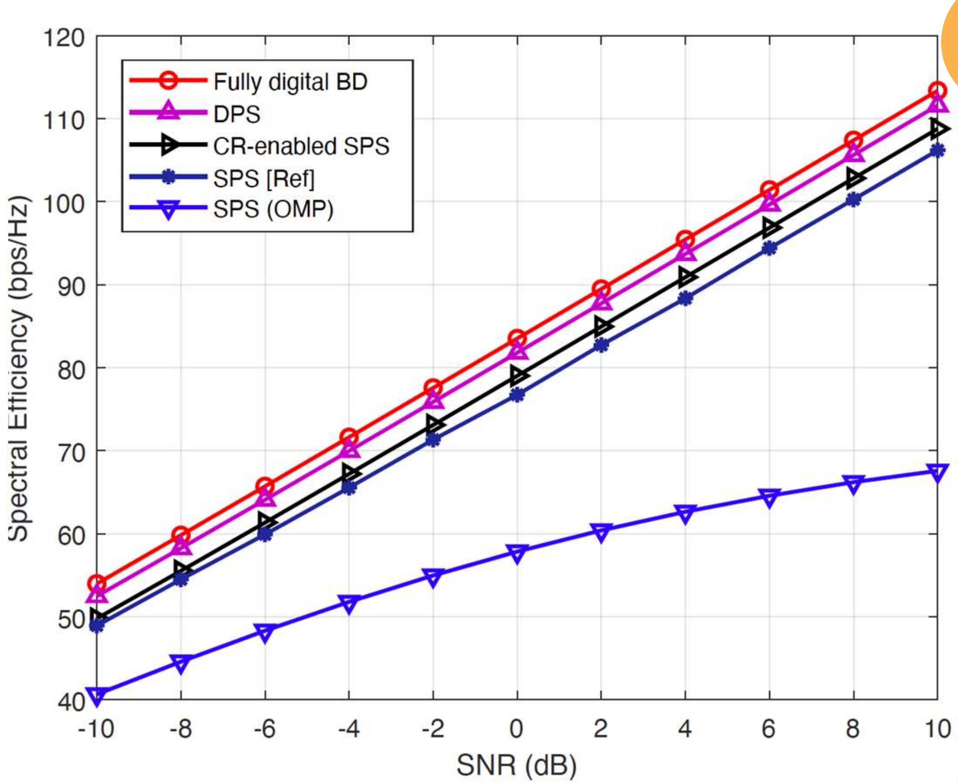


  - Modified K-means algorithm
    - Centroid: $\mathbf{x}_j^\star = \boldsymbol{\lambda}_1 \left(\sum_{i \in \mathcal{D}_j} \mathbf{y}_i \mathbf{y}_i^H \right)$
    - Clustering: $j^\star = \arg\max_j \quad \left| \mathbf{y}_i^H \mathbf{x}_j \right|^2$

Boost Computational Efficiency

❖ Simulation results (Fully-connected)

$N_t = 256,\ N_r = 16,\ K = 3,\ F = 128,\ N_s = 3,\ N_{RF}^{t} = 9, \text{ and } N_{RF}^{r} = 3$



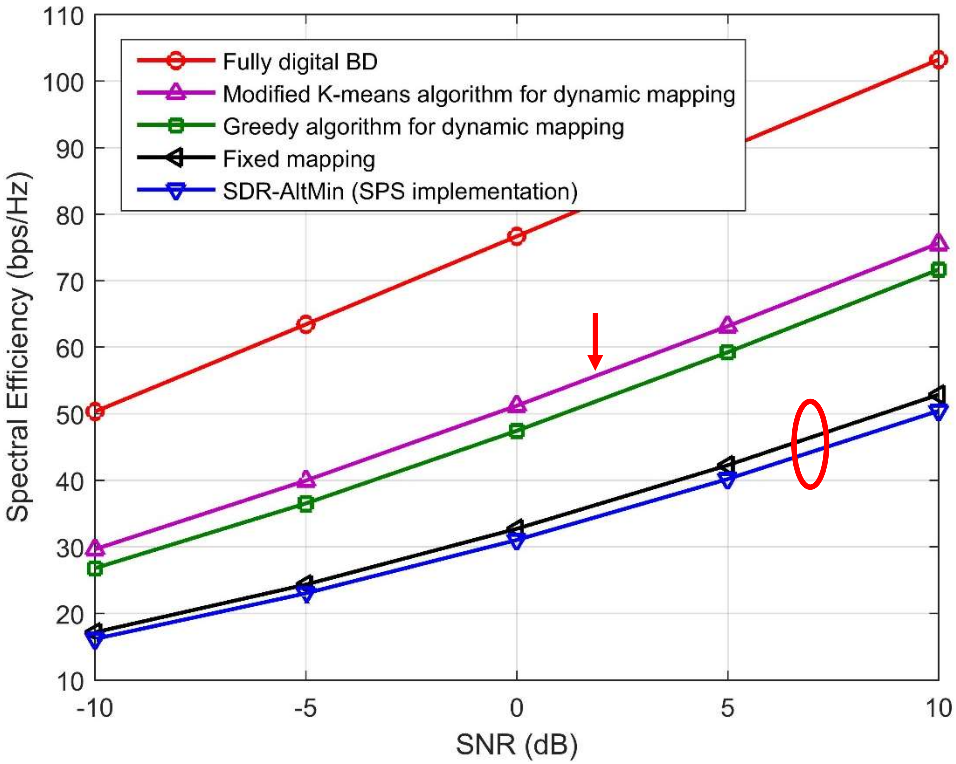
- Achieve near-optimal spectral efficiency and optimal multiplexing gain with low-complexity algorithms
- Effectiveness of the proposed CR-enabled SPS method

[Ref] F. Sohrabi and W. Yu, "Hybrid Analog and Digital Beamforming for mmWave OFDM Large-Scale Antenna Arrays," *IEEE J. Sel. Areas Commun.*, vol. 35, no. 7, pp. 1432-1443, July 2017.

Boost Computational Efficiency

❖ Simulation results (Partially-connected)

$N_{\mathrm{t}} = 256,\ N_{\mathrm{r}} = 16,\ K = 4,\ F = 128,\ N_s = 2\quad N_{\mathrm{RF}}^{\mathrm{t}} = KN_s$, and $N_{\mathrm{RF}}^{\mathrm{r}} = N_s$



- Simply doubling PSs in the partially-connected mapping is far from satisfactory
- Superiority of the modified K-means algorithm with lower computational complexity than the greedy algorithm

Boost Computational Efficiency

❖ Discussions

➢ Comparison of computational complexity

Imple-mentation	Structure	Design approach	Hardware complexity (No. of phase shifters)	Computational complexity	Performance
SPS	Fully-connected	MO-AltMin	$N_{\mathrm{RF}}^{\mathrm{t}} N_{\mathrm{t}}$	Extremely high	✓✓✓
	Partially-connected	SDR-AltMin	N_{t}	High	✓
DPS	Fully-connected	Matrix decomposition	$2N_{\mathrm{RF}}^{\mathrm{t}}(N_{\mathrm{t}} - N_{\mathrm{RF}}^{\mathrm{t}})$	$\mathcal{O}\left({N_{\mathrm{RF}}^{\mathrm{t}}}^2 N_{\mathrm{t}} F\right)$	✓✓✓✓
	Partially-connected	Modified K-means	$2N_{\mathrm{t}}$	$\mathcal{O}\left(N {N_{\mathrm{RF}}^{\mathrm{t}}}^2 N_{\mathrm{t}} F\right)$	✓✓

➢ The proposed DPS implementation enables low-complexity design for hybrid beamforming

Boost Computational Efficiency

- Discussions
  - The number of RF chains has been reduced to the minimum

$$N_{\mathrm{RF}}^{\mathrm{t}} = K N_s$$

  - A large number of high-precision phase shifters are still needed

	Fully-connected	Partially-connected
SPS	$N_t N_{RF}$	N_t
DPS	$2N_t N_{RF}$	$2N_t$

  - Need to adapt the phases to channel states

- Practical phase shifters are typically with coarsely quantized phases

How to reduce # phase shifters?

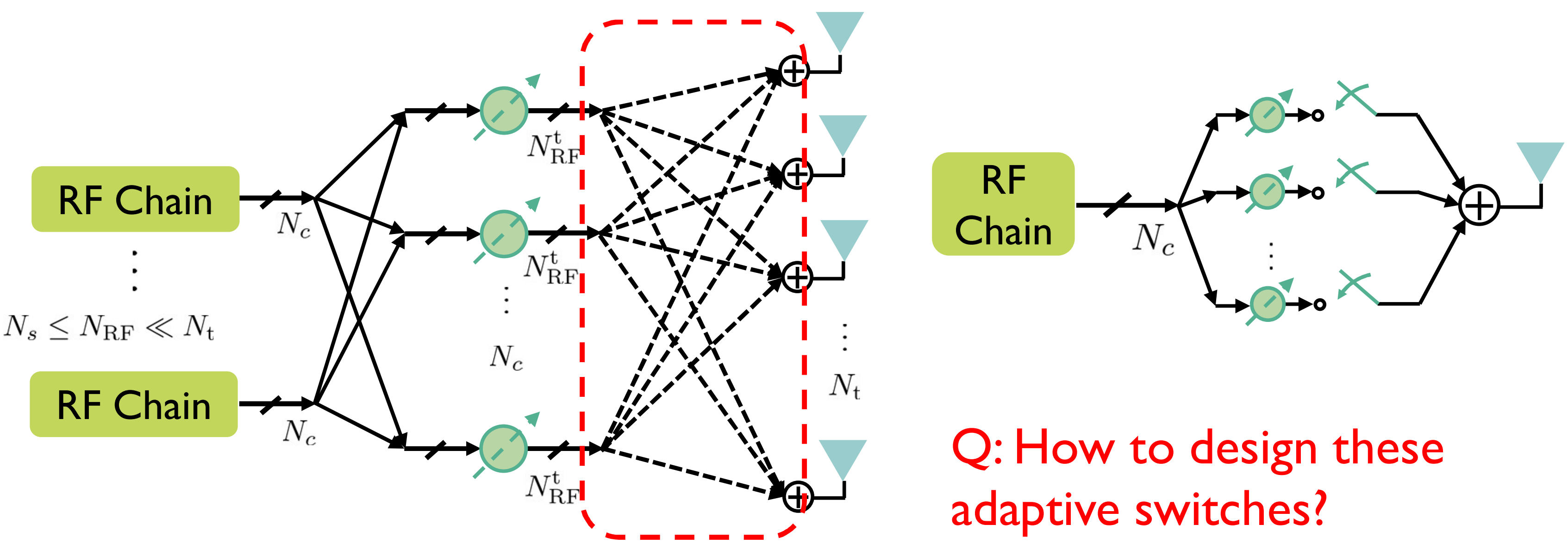
Fight for Hardware Efficiency: How Many Phase Shifters Are Needed?

[Ref] X. Yu, J. Zhang, and K. B. Letaief, "Hybrid precoding in millimeter wave systems: How many phase shifters are needed?" in *Proc. IEEE Global Commun. Conf. (Globecom)*, Singapore, Dec. 2017. **(Best Paper Award)**

[Ref] X. Yu, J. Zhang, and K. B. Letaief, "A hardware-efficient analog network structure for hybrid precoding in millimeter wave systems," *IEEE J. Sel. Topics Signal Process., Special Issue on Hybrid Analog-Digital Signal Processing for Hardware-Efficient Large Scale Antenna Arrays*, vol. 12, no. 2, pp. 282-297, May 2018.

Fight for Hardware Efficiency

❖ Fixed phase shifter (FPS) implementation



Q: How to design these adaptive switches?

➢ N_c multi-channel fixed PSs [Z. Feng *et al.*, 2014]

Fight for Hardware Efficiency

- Problem formulation
  - $\mathcal{A}_x$: $\mathbf{F}_{\mathrm{RF}} = \mathbf{SC}$
  - FPS matrix $\mathbf{C} = \mathrm{diag}(\overbrace{\mathbf{c}, \mathbf{c}, \cdots, \mathbf{c}}^{N_{\mathrm{RF}}^{\mathrm{t}}})$, $\quad \mathbf{c} = \frac{1}{\sqrt{N_c}}\left[e^{\jmath\theta_1}, e^{\jmath\theta_2}, \cdots, e^{\jmath\theta_{N_c}}\right]^T$
  - Binary switch matrix $\mathbf{S} \in \{0,1\}^{N_{\mathrm{t}} \times N_c N_{\mathrm{RF}}^{\mathrm{t}}}$

Phases are fixed

$$\begin{array}{ll} \underset{\mathbf{S}, \mathbf{F}_{\mathrm{BB}}}{\text{minimize}} & \|\mathbf{F}_{\mathrm{opt}} - \mathbf{SCF}_{\mathrm{BB}}\|_F^2 \\ \text{subject to} & \mathbf{S} \in \{0,1\}^{N_{\mathrm{t}} \times N_c N_{\mathrm{RF}}^{\mathrm{t}}} \end{array}$$

NP-hard

- An objective upper bound enables a low-complexity algorithm
  - Enforce a semi-orthogonal constraint on $\mathbf{F}_{\mathrm{BB}}$ [X.Yu *et al.*, 2016]

$$\mathbf{F}_{\mathrm{BB}}^H \mathbf{F}_{\mathrm{BB}} = \alpha^2 \mathbf{F}_{\mathrm{DD}}^H \mathbf{F}_{\mathrm{DD}} = \alpha^2 \mathbf{I}_{K N_s}$$

$$\|\mathbf{F}_{\mathrm{opt}} - \mathbf{SCF}_{\mathrm{BB}}\|_F^2 \leq \|\mathbf{F}_{\mathrm{opt}}\|_F^2 - 2\alpha \Re \,\mathrm{Tr}\left(\mathbf{F}_{\mathrm{DD}} \mathbf{F}_{\mathrm{opt}}^H \mathbf{SC}\right) + \alpha^2 \|\mathbf{S}\|_F^2$$

Fight for Hardware Efficiency

- Alternating minimization
  - Digital precoder

$$\begin{aligned} \underset{\mathbf{F}_{\mathrm{DD}}}{\text{maximize}} \quad & \Re \operatorname{Tr}\left(\mathbf{F}_{\mathrm{DD}} \mathbf{F}_{\mathrm{opt}}^{H} \mathbf{S} \mathbf{C}\right) \\ \text{subject to} \quad & \mathbf{F}_{\mathrm{DD}}^{H} \mathbf{F}_{\mathrm{DD}} = \mathbf{I}_{K N_s} \end{aligned}$$

  - Semi-orthogonal Procrustes solution $\mathbf{F}_{\mathrm{DD}} = \mathbf{V}_1 \mathbf{U}^H$

$$\alpha \mathbf{F}_{opt}^{H} \mathbf{S} \mathbf{C} = \mathbf{U} \boldsymbol{\Sigma} \mathbf{V}_1^{H}$$

  - Switch matrix optimization

$$\begin{aligned} \underset{\alpha, \mathbf{S}}{\text{minimize}} \quad & \left\| \Re\left(\mathbf{F}_{\mathrm{opt}} \mathbf{F}_{\mathrm{DD}}^{H} \mathbf{C}^{H}\right) - \alpha \mathbf{S} \right\|_F^2 \\ \text{subject to} \quad & \mathbf{S} \in \{0,1\}^{N_{\mathrm{t}} \times N_c N_{\mathrm{RF}}^{\mathrm{t}}} \end{aligned}$$

  - Once α is optimized, the optimal $\mathbf{S}$ is determined correspondingly

$$\mathbf{S}^{\star} = \begin{cases} \mathbb{1}\left\{ \Re\left(\mathbf{F}_{\mathrm{opt}} \mathbf{F}_{\mathrm{DD}}^{H} \mathbf{C}^{H}\right) > \frac{\alpha}{2} \mathbf{1}_{N_{\mathrm{t}} \times N_c N_{\mathrm{RF}}^{\mathrm{t}}} \right\} & \alpha > 0 \\ \mathbb{1}\left\{ \Re\left(\mathbf{F}_{\mathrm{opt}} \mathbf{F}_{\mathrm{DD}}^{H} \mathbf{C}^{H}\right) < \frac{\alpha}{2} \mathbf{1}_{N_{\mathrm{t}} \times N_c N_{\mathrm{RF}}^{\mathrm{t}}} \right\} & \alpha < 0 \end{cases}$$

Fight for Hardware Efficiency

- Alternating minimization (cont.)
  - Optimization of α

$$\alpha^{\star} = \arg \min_{\{\tilde{x}_i, \bar{x}_i\}_{i=1}^n} \{f(\tilde{x}_i), f(\bar{x}_i)\}$$

$$\tilde{\mathbf{x}} = \mathrm{vec}(\Re(\mathbf{F}_{\mathrm{opt}}\mathbf{F}_{\mathrm{DD}}^H\mathbf{C}^H)) \qquad \bar{x}_i \triangleq \begin{cases} \frac{\sum_{j=1}^i \tilde{x}_j}{i} & \alpha < 0 \text{ and } \frac{\sum_{j=1}^i \tilde{x}_j}{i} \in [2\tilde{x}_i, 2\tilde{x}_{i+1}] \\ \frac{\sum_{j=i+1}^n \tilde{x}_j}{n-i} & \alpha > 0 \text{ and } \frac{\sum_{j=i+1}^n \tilde{x}_j}{n-i} \in [2\tilde{x}_i, 2\tilde{x}_{i+1}] \\ +\infty & \text{otherwiese} \end{cases}$$

$$\tilde{\mathbf{x}} \in \mathbb{R}^n, \quad n = N_{\mathrm{t}}N_{\mathrm{RF}}^{\mathrm{t}}N_c$$

  - Search dimension: $|\mathcal{X}| = 2N_{\mathrm{t}}N_{\mathrm{RF}}^{\mathrm{t}}N_c$ ✖
  - Acceleration: Optimal point can only be obtained at $\bar{x}_i$

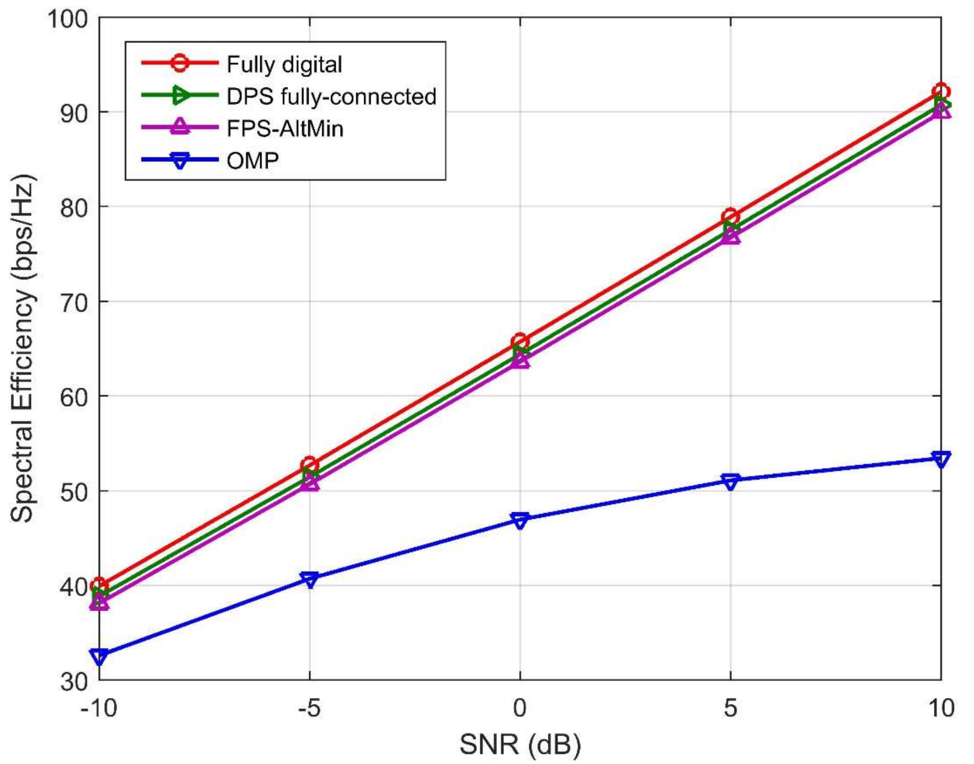
$$\alpha^{\star} = \arg \min_{\bar{x}_i} \quad f(\bar{x}_i)$$

  - Search dimension $\ll 2N_{\mathrm{t}}N_{\mathrm{RF}}^{\mathrm{t}}N_c$
  - Convergence guarantee

Fight for Hardware Efficiency

❖ Simulation results: MU-MC systems

$N_{\mathrm{t}} = 144,\ N_{\mathrm{r}} = 16,\ K = 4,\ F = 128,\ N_s = 2,\ N_{\mathrm{RF}}^{\mathrm{t}} = 8$, and $N_{\mathrm{RF}}^{\mathrm{r}} = 2$

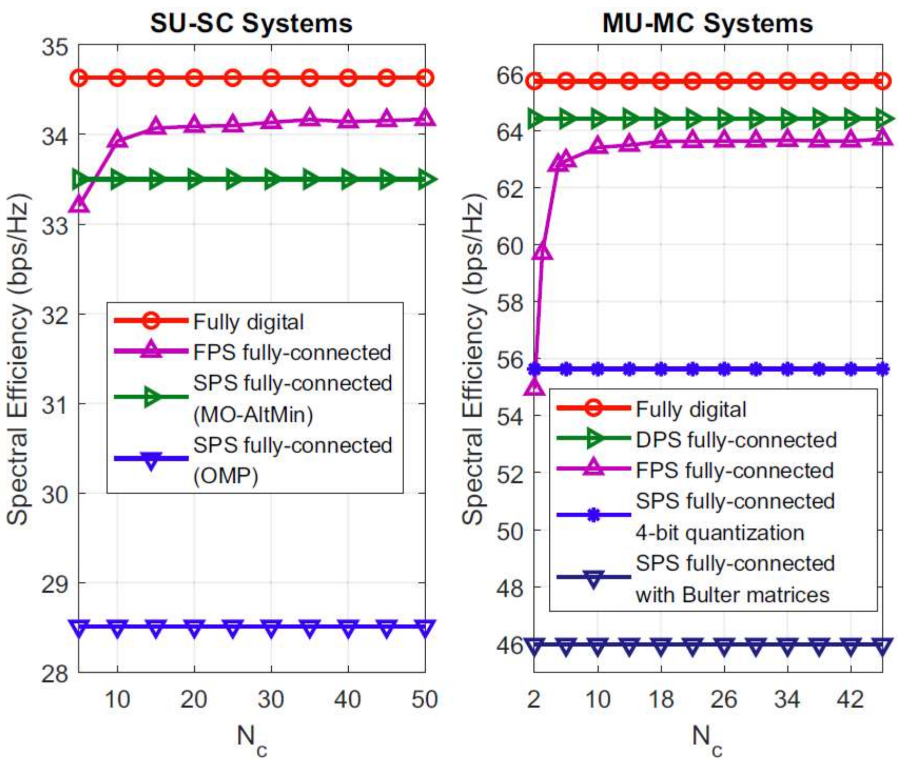


- Slightly inferior to the DPS fully-connected mapping with much fewer PSs
- Significant improvement over the OMP algorithm

Fight for Hardware Efficiency

❖ Simulation results: How many PSs are needed?

$N_{\mathrm{t}} = 256,\ N_{\mathrm{r}} = 16,\ K = 4,\ F = 128,\ N_s = 2,\ N_{\mathrm{RF}}^{\mathrm{t}} = 8$, and $N_{\mathrm{RF}}^{\mathrm{r}} = 2$



- Only ~10 fixed phase shifters are sufficient!
- 200 times reduction compared with the DPS implementation

Fight for Hardware Efficiency

- Simulation results: How much power can be saved?

$N_t = 256,\ N_r = 16,\ K = 4,\ F = 128,\ N_s = 2,\ N_{RF}^{t} = 8$, and $N_{RF}^{r} = 2$

TABLE II
POWER CONSUMPTION OF THE ANALOG NETWORK FOR DIFFERENT HYBRID PRECODER STRUCTURES IN MU-MC SYSTEMS

	Phase shifter		Other hardware		Total power‡
	Number N_{PS}	Type	Hardware	Number N_{OC}	P_{total}
DPS fully-connected	2304	Adaptive	N/A	N/A	115.2 W
FPS fully-connected	10	Fixed§	Switch	11520	59.2 W
SPS fully-connected 4-bit quantization	1152	Adaptive	N/A	N/A	57.6 W
FPS fully-connected	2	Fixed	Switch	2304	11.84 W
SPS fully-connected with Butler matrices	3456	Fixed	Coupler	4032	109.44 W

Conclusions

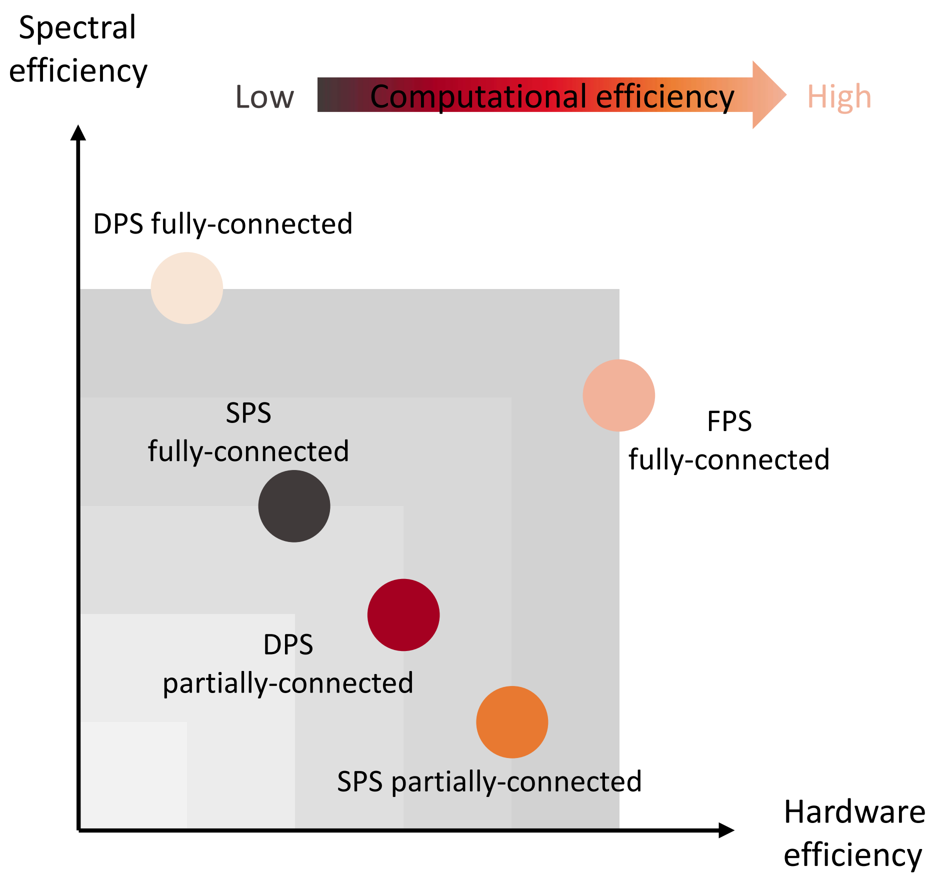
Conclusions

- Questions answered
  - **Q1:** Can hybrid beamforming provide performance close to the fully digital one? YES
  - **Q2:** How many RF chains are needed? KN_s
  - **Q3:** How many phase shifters are needed? ~10 FPSs
  - **Q4:** How to efficiently design hybrid beamforming algorithms?

    Alternating minimization provides the basic principle
    Manifold optimization provides good benchmark
    Convex relaxation enables low-complexity algorithms

Conclusions

❖ Comparisons between different hybrid precoder structures



- **SPS**: May not be a good choice
- **DPS**: An excellent candidate for low-complexity algorithms
- **FPS**: Trade-off between the hardware and computational complexity, with satisfactory performance

Conclusions

❖ Our own results

- **X. Yu**, J.-C. Shen, **J. Zhang**, and K. B. Letaief, "Alternating minimization algorithms for hybrid precoding in millimeter wave MIMO systems," *IEEE J. Sel. Topics Signal Process., Special Issue on Signal Process. for Millimeter Wave Wireless Commun.*, vol. 10, no. 3, pp. 485-500, Apr. 2016. (**The 2018 SPS Young Author Best Paper Award**)
- **J. Zhang**, **X. Yu**, and K. B. Letaief, "Hybrid beamforming for 5G and beyond millimeter-wave systems: A holistic view," *IEEE Open J. Commun. Society*, vol. 1, no. 1, pp. 77–91, Dec. 2019.
- **X. Yu**, **J. Zhang,** and K. B. Letaief, "Doubling phase shifters for efficient hybrid precoding in millimeter-wave multiuser OFDM systems," *J. Commun. Inf. Netw.*, vol. 4, no. 2, pp. 51-67, Jul. 2019.
- **X. Yu**, **J. Zhang**, and K. B. Letaief, "Hybrid precoding in millimeter wave systems: How many phase shifters are needed?" in *Proc. IEEE Global Commun. Conf. (Globecom)*, Singapore, Dec. 2017. **(Best Paper Award)**
- **X. Yu**, **J. Zhang**, and K. B. Letaief, "A hardware-efficient analog network structure for hybrid precoding in millimeter wave systems," *IEEE J. Sel. Topics Signal Process., Special Issue on Hybrid Analog-Digital Signal Processing for Hardware-Efficient Large Scale Antenna Arrays*, vol. 12, no. 2, pp. 282-297, May 2018.

Conclusions

❖ References in this webinar

- X. Zhang, A. F. Molisch, and S.-Y. Kung, “Variable-phase-shift-based RF-baseband codesign for MIMO antenna selection,” *IEEE Trans. Signal Process.*, vol. 53, no. 11, pp. 4091-4103, Nov. 2005.
- E. Zhang and C. Huang, “On achieving optimal rate of digital precoder by RF-baseband codesign for MIMO systems,” in *Proc. IEEE 80th Veh. Technol. Conf. (VTC2014-Fall)*, Vancouver, BC, Sep. 2014, pp. 1-5.
- O. E. Ayach, S. Rajagopal, S. Abu-Surra, Z. Pi, and R. W. Heath, Jr., “Spatially sparse precoding in millimeter wave MIMO systems,” *IEEE Trans. Wireless Commun.*, vol. 13, no. 3, pp. 1499-1513, Mar. 2014.
- N. Boumal, B. Mishra, P. A. Absil, R. Sepulchre, “Manopt, a Matlab toolbox for optimization on manifolds,” *The Journal of Machine Learning Research* vol. 15, no, 1, pp. 1455-1459, 2014.
- X. Gao, L. Dai, S. Han, C.-L. I, and R. W. Heath, Jr., “Energy-efficient hybrid analog and digital precoding for mmWave MIMO systems with large antenna arrays,” *IEEE J. Sel. Areas in Commun.*, vol. 34, no. 4, pp. 998-1009, Apr. 2016.

Conclusions

❖ References in this webinar

- F. Sohrabi and W. Yu, “Hybrid digital and analog beamforming design for large-scale antenna arrays,” *IEEE J. Sel. Topics Signal Process.*, vol. 10, no. 3, pp. 501-513, Apr. 2016.
- Y. Lee, C.-H. Wang, and Y.-H. Huang, “A hybrid RF/baseband precoding processor based on parallel-index-selection matrix-inversion-bypass simultaneous orthogonal matching pursuit for millimeter wave MIMO systems,” *IEEE Trans. Signal Process.*, vol. 63, no. 2, pp. 305–317, Jan. 2015.
- L. Liang, W. Xu, and X. Dong, “Low-complexity hybrid precoding in massive multiuser MIMO systems,” *IEEE Wireless Commun. Lett.*, vol. 3, no. 6, pp. 653–656, Dec. 2014.
- F. Sohrabi and W. Yu, “Hybrid Analog and Digital Beamforming for mmWave OFDM Large-Scale Antenna Arrays,” *IEEE J. Sel. Areas Commun.*, vol. 35, no. 7, pp. 1432-1443, July 2017.
- Z. Feng, S. Fu, T. Ming, and D. Liu, “Multichannel continuously tunable microwave phase shifter with capability of frequency doubling,” *IEEE Photon. J.,* vol. 6, no. 1, pp. 1–8, Feb. 2014.

Conclusions

❖ References in this webinar

- A. Alkhateeb, O. El Ayach, G. Leus, and R. W. Heath, Jr., "Channel estimation and hybrid precoding for millimeter wave cellular systems," *IEEE J. Sel. Topics Signal Process*., vol. 8, no. 5, pp. 831-846, Oct. 2014.
- Z. Gao, C. Hu, L. Dai, and Z. Wang, "Channel estimation for millimeter-wave massive MIMO with hybrid precoding over frequency-selective fading channels," *IEEE Commun. Lett*., vol. 20, no. 6, pp. 1259-1262, June 2016.
- A. Alkhateeb, O. El Ayach, G. Leus, and R. W. Heath, Jr., "Hybrid precoding for millimeter wave cellular systems with partial channel knowledge," in *Proc. Inf. Theory Applications Workshop (ITA)*, San Diego, CA, 2013, pp. 1-5.
- M. N. Kulkarni, A. Ghosh, and J. G. Andrews, "A comparison of MIMO techniques in downlink millimeter wave cellular networks with hybrid beamforming," *IEEE Trans. Commun*., vol. 64, no. 5, pp. 1952-1967, May 2016.
- X. Gao, L. Dai, Y. Sun, S. Han, and C.-L. I, "Machine learning inspired energy-efficient hybrid precoding for mmWave massive MIMO systems," in *Proc. IEEE Int. Conf. Commun. (ICC),* Paris, France, May, 2017, pp. 1-6.

Conclusions

❖ References in this webinar

- A. Alkhateeb, S. Alex, P. Varkey, Y. Li, Q. Qu, and D. Tujkovic, "Deep learning coordinated beamforming for highly-mobile millimeter wave systems," *IEEE Access*, vol. 6, pp. 37328-37348, 2018.
- W. Roh et al., "Millimeter-wave beamforming as an enabling technology for 5G cellular communications: theoretical feasibility and prototype results," *IEEE Commun. Mag.*, vol. 52, no. 2, pp. 106-113, Feb. 2014.
- J. Mo, A. Alkhateeb, S. Abu-Surra and R. W. Heath, "Hybrid architectures with few-bit ADC receivers: Achievable rates and energy-rate tradeoffs," *IEEE Trans. Wireless Commun.*, vol. 16, no. 4, pp. 2274-2287, Apr. 2017.
- Y. Zeng and R. Zhang, "Millimeter Wave MIMO With Lens Antenna Array: A New Path Division Multiplexing Paradigm," *IEEE Trans. Commun.*, vol. 64, no. 4, pp. 1557-1571, Apr. 2016.
- X. Gao, L. Dai, S. Han, C.-L. I, and X. Wang, "Reliable beamspace channel estimation for millimeter-wave massive MIMO systems with lens antenna array," *IEEE Trans. Wireless Commun.*, vol. 16, no. 9, pp. 6010-6021, Sep. 2017.

Conclusions

❖ Other references

- A. F. Molisch, V. V. Ratnam, S. Han, Z. Li, S. L. H. Nguyen, L. Li, and K. Haneda, "Hybrid beamforming for massive MIMO: A survey," *IEEE Commun. Mag.*, vol. 55, no. 9, pp. 134–141, Sep. 2017.
- J. Brady, N. Behdad, and A. M. Sayeed, "Beamspace MIMO for millimeter-wave communications: System architecture, modeling, analysis, and measurements," *IEEE Trans. Antennas Propag.*, vol. 61, no. 7, pp. 3814–3827, Jul. 2013.
- J. Singh and S. Ramakrishna, "On the feasibility of codebook-based beamforming in millimeter wave systems with multiple antenna arrays," *IEEE Trans. Wireless Commun.*, vol. 14, no. 5, pp. 2670–2683, May 2015.
- S. Park, A. Alkhateeb, and R.W. Heath, Jr., "Dynamic subarrays for hybrid precoding in wideband mmWave MIMO systems," *IEEE Trans. Wireless Commun.*, vol. 16, no. 5, pp. 2907–2920, May 2017.
- T. E. Bogale, L. B. Le, A.Haghighat, and L.Vandendorpe, "On the number of RF chains and phase shifters, and scheduling design with hybrid analog digital beamforming," *IEEE Trans. Wireless Commun.*, vol. 15, no. 5, pp. 3311–3326, May 2016.

Conclusions

❖ Other references

- A. Alkhateeb and R.W. Heath, Jr., "Frequency selective hybrid precoding for limited feedback millimeter wave systems," *IEEE Trans. Commun.*, vol. 64, no. 5, pp. 1801–1818, May 2016.
- R. Mendez-Rial, C. Rusu, N. Gonzalez-Prelcic, A. Alkhateeb, and R. W. Heath, Jr., "Hybrid MIMO architectures for millimeter wave communications: Phase shifters or switches?" *IEEE Access*, vol. 4, pp. 247–267, Jan. 2016.
- W. Ni and X. Dong, "Hybrid block diagonalization for massive multiuser MIMO systems," *IEEE Trans. Commun.*, vol. 64, no. 1, pp. 201–211, Jan. 2016.
- T. E. Bogale and L. B. Le, "Beamforming for multiuser massive MIMO systems: Digital versus hybrid analog-digital," in *Proc. IEEE Global Commun. Conf.*, Austin, TX, USA, Dec. 2014, pp. 4066–4071.

Thanks

For more information and Matlab codes:

https://yuxianghao.github.io/

http://www.eie.polyu.edu.hk/~jeiezhang